

QUANTUM PRINCIPAL BUNDLES OVER QUANTUM REAL PROJECTIVE SPACES

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ABSTRACT. Two hierarchies of quantum principal bundles over quantum real projective spaces are constructed. One hierarchy contains bundles with $U(1)$ as a structure group, the other has the quantum group $SU_q(2)$ as a fibre. Both hierarchies are obtained by the process of prolongation from bundles with the cyclic group of order 2 as a fibre. The triviality or otherwise of these bundles is determined by using a general criterion for a prolongation of a comodule algebra to be a cleft Hopf-Galois extension.

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1. INTRODUCTION

For considerable time it has been argued that faithfully flat Hopf-Galois extensions or Hopf-Galois extensions admitting strong connections or *principal comodule algebras* should be considered as objects representing principal bundles in noncommutative geometry; see e.g. [20], [3], [9], [10]. Prolongation of the structure group to a larger group is one of standard methods of constructing principal bundles in classical geometry. In the case of principal comodule algebras an analogous (or dual) construction starts with a principal comodule algebra over a Hopf algebra \bar{H} and produces a principal comodule algebra over any Hopf algebra H for which there exists a Hopf algebra map $H \rightarrow \bar{H}$, by using the cotensor product; see [20, Remark 3.11]. In this paper we derive necessary and sufficient conditions for a prolongation to be trivial, and study prolongations of quantum (Euclidean) spheres.

Although this text deals primarily with prolongations we also comment on the opposite process of *reduction*. This is the process by which a principal comodule H -algebra is

reduced to a principal \bar{H} -comodule algebra along a surjective Hopf algebra map $H \rightarrow \bar{H}$. In particular we illustrate the possibility of reducing trivial (that is smash product) principal comodule algebras to non-trivial principal comodule algebras.

The paper is organised as follows. In Section 2 we gather preliminary information about principal comodule algebras and prolongations. In Section 3 we present a hierarchy (or a sequence of surjective maps) of coordinate algebras of quantum spheres. Each algebra in this hierarchy is a principal comodule algebra of the Hopf algebra $\mathcal{O}(\mathbb{Z}_2)$ generated by a single self-adjoint element that squares to 1. Invariant subalgebras can be identified with quantum real projective spaces. We spend some time presenting the algebraic structure of low dimensional quantum spheres and of the coordinate algebra of the quantum real projective plane $\mathcal{O}(\mathbb{RP}_q^2)$. The main results of the paper are contained in Sections 4 and 5. In the former, a sufficient and necessary condition for prolongation of a comodule algebra to be cleft is derived. In the latter we prolong quantum spheres to principal comodule algebras of $\mathcal{O}(U(1))$ and $\mathcal{O}(SU_q(2))$. We use criterion derived in Section 4 to determine which of the constructed comodule algebras are trivial, and we describe algebraic contents of algebras obtained as prolongations. In particular, we prove that $\mathcal{O}(U(1))$ -prolongations of coordinate algebras of quantum spheres $\mathcal{O}(S_q^m)$, $m > 1$, are non-trivial. Furthermore, we give the presentation by generators and relations of $\mathcal{O}(U(1))$ -prolongations of even-dimensional spheres, describe their irreducible $*$ -representations and construct Fredholm modules over them. We prove that $\mathcal{O}(U(1))$ -prolongations of odd-dimensional spheres are isomorphic as algebras (but not as comodules) to $\mathcal{O}(S_q^{2n+1}) \otimes \mathcal{O}(U(1))$. We describe the smash product structure of the $\mathcal{O}(SU_q(2))$ -prolongation of $\mathcal{O}(S_q^2)$ and explain that, for $m = 2, 3$, the algebras $\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ are non-trivial principal $\mathcal{O}(U(1))$ -comodule algebras obtained by reduction of trivial principal $\mathcal{O}(SU_q(2))$ -comodule algebras.

Notation. All algebras in this paper are over the field of complex numbers. They are associative, unital and $*$ -algebras (the latter minor assumption might be dropped and then the choice of the ground field may be freed). Hopf algebras are assumed to have bijective antipodes. The comultiplication in a Hopf algebra H is denoted by Δ , counit by ε and the antipode by S . A subscript is used sometimes if more than one Hopf algebra appears. We use the standard Sweedler notation for comultiplication, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (summation implicit), for all $h \in H$. Whenever needed the coaction on a right H -comodule M is denoted by ρ^H , and the Sweedler notation $\rho^H(m) = m_{(0)} \otimes m_{(1)}$ is used. The vector space of right H -colinear maps from M to N is denoted by $\text{Hom}^H(M, N)$. By a (right) H -comodule algebra we mean an algebra and a right H -comodule with a coaction that is an algebra map. By σ we denote the flip map between vector spaces, $\sigma : V \otimes W \rightarrow W \otimes V$, $v \otimes w \mapsto w \otimes v$.

2. PRINCIPAL COMODULE ALGEBRAS

Principal comodule algebras are simply the same as faithfully flat Hopf-Galois extensions (by a Hopf algebra with bijective antipode). The definition of a principal comodule algebra can equivalently be formulated in terms of *strong connections*.

DEFINITION 2.1 (see e.g. [2], [10]). Let H be a Hopf algebra with bijective antipode, and let A be a right H -comodule algebra with coaction ρ^H , multiplication $\mu : A \otimes A \rightarrow A$ and

unit $\eta : \mathbb{C} \rightarrow A$. A map

$$\ell : H \longrightarrow A \otimes A$$

is called a *strong connection* if

$$(2.1a) \quad \ell(1) = 1 \otimes 1,$$

$$(2.1b) \quad \mu \circ \ell = \eta \circ \varepsilon,$$

$$(2.1c) \quad (\ell \otimes \text{id}) \circ \Delta = (\text{id} \otimes \rho^H) \circ \ell,$$

$$(2.1d) \quad (S \otimes \ell) \circ \Delta = (\sigma \otimes \text{id}) \circ (\rho^H \otimes \text{id}) \circ \ell.$$

If an H -comodule algebra A admits a strong connection, then it is called a *principal H -comodule algebra*.

REMARK 2.2. Let us comment how the existence of a strong connection is related to the Hopf-Galois condition. Denote by

$$B = A^{\text{co}H} := \{b \in A \mid \rho^H(b) = b \otimes 1\}$$

the subalgebra of H -coaction invariants or H -coinvariants. Consider the map

$$\text{can} : A \otimes_B A \rightarrow A \otimes H, \quad p \otimes q \longmapsto pq_{(0)} \otimes q_{(1)},$$

called the *canonical map*. A is called a *Hopf-Galois extension* of B , provided the canonical map is bijective. Denote by $\psi : A \otimes A \rightarrow A \otimes_B A$ the natural surjection. If A is a principal extension, then the map can is invertible, and the inverse can be written explicitly as

$$\text{can}^{-1} : p \otimes h \longmapsto p\psi(\ell(h)).$$

Hence $\psi(\ell(h)) = \text{can}^{-1}(1 \otimes h)$. While the strong connection is non-unique, its projection on the tensor product over B , $\psi \circ \ell$ (called the *translation map*), is.

Geometrically, one should understand principal comodule algebras as coordinate algebras of quantum principal bundles. From this point of view H is the algebra of functions on the fibre (structure quantum group) and the algebra of functions on the base is identified with the coaction invariant subalgebra. This intuitive point of view can also be argued categorically using synthetic approach to noncommutative geometry [1].

Cleft extensions are examples of principal comodule algebras. These are principal comodule algebras for which there exists a right H -colinear map $j : H \rightarrow A$ that is unital, i.e. $j(1) = 1$ and convolution invertible, i.e. there exists a map $j^{-1} : H \rightarrow A$ such that, for all $h \in H$,

$$j(h_{(1)})j^{-1}(h_{(2)}) = j^{-1}(h_{(1)})j(h_{(2)}) = \varepsilon(h)1.$$

The map j is referred to as a *cleaving map*. In this case a strong connection can be defined as the composite

$$(2.2) \quad \ell = (j^{-1} \otimes j) \circ \Delta.$$

Equivalently, cleft extensions can be characterised as those principal H -comodule algebras that are isomorphic to $A^{\text{co}H} \otimes H$ as left $A^{\text{co}H}$ -modules and right H -comodules (the right H -coaction on $A^{\text{co}H} \otimes H$ is $\text{id} \otimes \Delta$). The isomorphism $\theta : A \rightarrow A^{\text{co}H} \otimes H$, $a \mapsto a_{(0)}j^{-1}(a_{(1)}) \otimes h$ induces a twisted tensor product or a crossed product with invertible cocycle algebra structure on $A^{\text{co}H} \otimes H$ [6].

Let A be a right H -comodule algebra. If there exists a right H -colinear algebra map $j : H \rightarrow A$, then j is convolution invertible with $j^{-1} = j \circ S$, so A is a cleft principal comodule algebra. In this case, the algebra A is isomorphic to the smash product of A^{coH} with H (a crossed product with a trivial cocycle) [7], and it has a geometric meaning of a trivial quantum principal bundle. Thus we refer to such principal comodule algebras as *trivial principal comodule algebras*.

The main construction used in this paper is given in the following

LEMMA 2.3. *Let \bar{A} be a principal \bar{H} -comodule algebra, with a strong connection ℓ . Denote by $B = \bar{A}^{co\bar{H}}$ a subalgebra of \bar{H} -coaction invariant elements. Let $\pi : H \rightarrow \bar{H}$ be a Hopf algebra map and consider the cotensor product*

$$\bar{A} \square_{\bar{H}} H := \left\{ \sum_i a^i \otimes h^i \in A \otimes H \mid \sum_i a^i_{(0)} \otimes a^i_{(1)} \otimes h^i = \sum_i a^i \otimes \pi(h^i_{(1)}) \otimes h^i_{(2)} \right\}.$$

View $\bar{A} \square_{\bar{H}} H$ as a right H -comodule subalgebra of the tensor algebra $A \otimes H$ with the coaction $\text{id} \otimes \Delta_H$. Then $\bar{A} \square_{\bar{H}} H$ is a principal H -comodule algebra, with a strong connection defined as the composite

$$(\sigma \otimes \text{id} \otimes \text{id}) \circ (S \otimes (\ell \circ \pi) \otimes \text{id}) \circ (\text{id} \otimes \Delta_H) \circ \Delta_H : H \longrightarrow (\bar{A} \square_{\bar{H}} H) \otimes (\bar{A} \square_{\bar{H}} H).$$

The principal H -comodule algebra A constructed in Lemma 2.3 is known as a *prolongation* of \bar{A} . Obviously, one can talk of prolongations also in the case of comodule algebras (not necessarily principal). There is no guarantee, however, that the prolonged H -comodule algebra $\bar{A} \square_{\bar{H}} H$ be principal. One particular case in which $\bar{A} \square_{\bar{H}} H$ is principal even though \bar{A} is not necessarily so is discussed in Section 4. In general, by the standard coalgebra-theoretic arguments (the Hom-cotensor relations),

$$(2.3) \quad (\bar{A} \square_{\bar{H}} H)^{coH} \simeq \text{Hom}^H(\mathbb{C}, \bar{A} \square_{\bar{H}} H) \simeq \text{Hom}^{\bar{H}}(\mathbb{C}, \bar{A}) \simeq \bar{A}^{co\bar{H}},$$

so the coaction-invariant subalgebras of both original and prolonged comodule algebras are isomorphic to each other.

The process opposite to prolongation is known as *reduction*. The main result in this area is the Hopf-Galois Reduction Theorem; see [8], [19] and [12]. In this text we use one lemma, which can be viewed as a particular corollary of the Hopf-Galois Reduction Theorem.

LEMMA 2.4. *Let $\pi : H \rightarrow \bar{H}$ be a surjective Hopf algebra map (so that H is a left \bar{H} -comodule algebra with coaction $(\pi \otimes \text{id}) \circ \Delta_H$) such that H is a left principal \bar{H} -comodule algebra. Let \bar{A} be a right \bar{H} -comodule algebra. If $\bar{A} \square_{\bar{H}} H$ is a principal H -comodule algebra, then \bar{A} is a principal \bar{H} -comodule algebra.*

3. QUANTUM SPHERES AND REAL PROJECTIVE SPACES

The noncommutative or quantum (Euclidean) spheres were introduced in [17] (in dimension 2) and in [22], [18] (for all n). Let q be a real number, $0 < q < 1$. The coordinate algebra $\mathcal{O}(S_q^{2n+1})$ of the odd-dimensional quantum sphere is the unital complex $*$ -algebra with generators z_0, z_1, \dots, z_n subject to the following relations:

$$(3.1a) \quad z_i z_j = q z_j z_i \quad \text{for } i < j, \quad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j,$$

$$(3.1b) \quad z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^n z_m z_m^*, \quad \sum_{m=0}^n z_m z_m^* = 1.$$

The coordinate algebra $\mathcal{O}(S_q^{2n})$ of the even-dimensional quantum sphere is the unital complex $*$ -algebra with generators z_0, z_1, \dots, z_n and relations (3.1) supplemented with $z_n^* = z_n$.¹ All these quantum spheres are right comodule algebras over the Hopf algebra $\mathcal{O}(\mathbb{Z}_2)$ generated by a self-adjoint grouplike element u satisfying $u^2 = 1$ (thus u is also unitary). The coaction is defined on generators by

$$(3.2) \quad z_i \longmapsto z_i \otimes u.$$

The coordinate algebra of the quantum real projective space $\mathcal{O}(\mathbb{RP}_q^m)$ is defined as the $\mathcal{O}(\mathbb{Z}_2)$ -coaction invariant subalgebra of $\mathcal{O}(S_q^m)$.² Thanks to the second of relations (3.1b) (the radius relation), every quantum sphere $\mathcal{O}(S_q^{2n})$, $\mathcal{O}(S_q^{2n+1})$ admits a strong connection

$$(3.3) \quad \ell(u) = \sum_{i=0}^n z_i \otimes z_i^*.$$

Thus each of the quantum spheres $\mathcal{O}(S_q^m)$ is a principal $\mathcal{O}(\mathbb{Z}_2)$ -comodule algebra or a quantum principal bundle over the quantum real projective space \mathbb{RP}_q^m . In the case $m = 2$ this was proven in [9], where also the algebra $\mathcal{O}(\mathbb{RP}_q^2)$ was defined.

Quantum spheres form a hierarchy of right $\mathcal{O}(\mathbb{Z}_2)$ -comodule $*$ -algebras

$$(3.4) \quad \dots \xrightarrow{f_5} \mathcal{O}(S_q^5) \xrightarrow{f_4} \mathcal{O}(S_q^4) \xrightarrow{f_3} \mathcal{O}(S_q^3) \xrightarrow{f_2} \mathcal{O}(S_q^2) \xrightarrow{f_1} \mathcal{O}(S_q^1),$$

where each of the f_m is a surjective $*$ -algebra and right $\mathcal{O}(\mathbb{Z}_2)$ -colinear map defined on generators as follows. In the odd case

$$(3.5) \quad f_{2n-1} : \mathcal{O}(S_q^{2n}) \longrightarrow \mathcal{O}(S_q^{2n-1}), \quad z_i \longmapsto \begin{cases} z_i & \text{if } i \neq n \\ 0 & \text{if } i = n \end{cases}$$

In the even case

$$(3.6) \quad f_{2n} : \mathcal{O}(S_q^{2n+1}) \longrightarrow \mathcal{O}(S_q^{2n}), \quad z_i \longmapsto z_i.$$

Note that in this case both z_n and z_n^* are mapped to the same self-adjoint element, which is consistent with the algebraic relations, since n is the maximal number in the set indexing generators of $\mathcal{O}(S_q^{2n+1})$ (so that (3.1a) are preserved) and z_n is normal by (3.1b).

The last three members of hierarchy (3.4) are of particular interest, so some comments on them are now in order. In the lowest dimensional case, the relations (3.1) do not depend on the parameter q . The algebra $\mathcal{O}(S_q^1)$ is a commutative polynomial algebra generated by a unitary element, say v , hence it can be identified with the algebra of polynomials on

¹We have learnt of the possibility of presenting even and odd dimensional quantum spheres in a uniform way from [14].

²In [15] the C^* -algebras of continuous functions on \mathbb{RP}_q^m were defined and shown to be isomorphic to Cuntz-Krieger algebras associated to suitable directed graphs. By this means Hong and Szymański were also able to calculate the K -theory of quantum real projective spaces.

the circle, $\mathcal{O}(S^1)$, or the algebra of polynomials on the group $U(1)$, $\mathcal{O}(U(1))$. $\mathcal{O}(S^1)$ is a Hopf algebra, v is a grouplike element and there is an (obvious) Hopf algebra map

$$(3.7) \quad \pi_2 : \mathcal{O}(U(1)) = \mathcal{O}(S^1) \longrightarrow \mathcal{O}(\mathbb{Z}_2), \quad v \longmapsto u.$$

From now on we use u to denote the unitary (and self-adjoint) generator of $\mathcal{O}(\mathbb{Z}_2)$ and we use v to denote the unitary generator of $\mathcal{O}(S^1)$.

The one before the penultimate member of the hierarchy, $\mathcal{O}(S_q^3)$, is the coordinate algebra of the quantum group $SU_q(2)$; see [23]. In terms of generators $a := z_0$ and $b := z_1^*$ the relations (3.1) come out as:

$$ab = qba, \quad ab^* = qb^*a, \quad bb^* = b^*b, \quad aa^* + bb^* = 1, \quad a^*a + q^{-2}bb^* = 1.$$

The Hopf algebra structure is given by the matrix co-representation:

$$\begin{aligned} \Delta : \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix} &\longmapsto \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ -q^{-1}b^* \otimes 1 & a^* \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes -q^{-1}b^* & 1 \otimes a^* \end{pmatrix}, \\ S : \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix} &\longmapsto \begin{pmatrix} a^* & -q^{-1}b \\ b^* & a \end{pmatrix}, \quad \varepsilon : \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The composite $f_1 \circ f_2 : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(U(1))$ is given on generators by $a \mapsto v$, $b \mapsto 0$ and turns out to be a Hopf algebra map. Consequently, the composite

$$(3.8) \quad \pi = \pi_2 \circ f_1 \circ f_2 : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{O}(\mathbb{Z}_2), \quad a \longmapsto u, \quad b \longmapsto 0,$$

is a Hopf algebra map. Since π is a Hopf algebra map it makes $\mathcal{O}(SU_q(2))$ into a (cocentral) $\mathcal{O}(\mathbb{Z}_2)$ -bicomodule by composing π with the comultiplication. The right coaction coincides with the coaction (3.2). The map

$$(3.9) \quad \iota : \mathcal{O}(\mathbb{Z}_2) \longrightarrow \mathcal{O}(SU_q(2)), \quad 1 \longmapsto 1, \quad u \longmapsto a,$$

is a bilinear splitting of π .

The penultimate member of the hierarchy $\mathcal{O}(S_q^2)$ is the coordinate algebra of the quantum equatorial Podleś sphere $S_{\sqrt{q}, \infty}^2$ (note the square root in the parameter!) [17]. The map $i : \mathcal{O}(S_q^2) \rightarrow \mathcal{O}(SU_{\sqrt{q}}(2))$, $z_0 \mapsto -q^{-1}ab$, $z_1 \mapsto bb^*$, is a $*$ -algebra inclusion. Furthermore, $i(\mathcal{O}(S_q^2))$ is a left coideal of the Hopf algebra $\mathcal{O}(SU_{\sqrt{q}}(2))$, i.e. the quantum 2-sphere S_q^2 is a quantum homogeneous space of the quantum group $SU_{\sqrt{q}}(2)$.

The algebra of functions on the quantum projective space $\mathcal{O}(\mathbb{RP}_q^2)$ is generated by

$$P = q^{-2}z_1^2, \quad R = z_0^2, \quad T = q^{-1}z_1z_0.$$

Generators P and R satisfy the following relations [13]:

$$\begin{aligned} P &= P^*, \quad T^2 = qPR, \quad RT^* = qT(-q^2P + 1), \quad R^*T = q^{-1}T^*(-P + 1), \\ RR^* &= q^6P^2 - q^2(1 + q^2)P + 1, \quad R^*R = q^{-2}P^2 - (1 + q^{-2})P + 1, \\ TT^* &= -q^2P^2 + P, \quad T^*T = q^{-2}(P - P^2), \\ RP &= q^4PR, \quad RT = q^2TR, \quad PT = q^{-2}TP. \end{aligned}$$

Equivalently, the coordinate algebra of the quantum projective space $\mathcal{O}(\mathbb{RP}_q^2)$ is a $*$ -algebra generated by P , R and T satisfying above relations (note that the squaring of q is needed to synchronise our conventions with those of [13]).

4. CLEFT PROLONGATIONS OF COMODULE ALGEBRAS

The aim of this section is to determine necessary and sufficient conditions for a prolongation of a right comodule algebra to be cleft. Throughout this section $\pi : H \rightarrow \bar{H}$ is a Hopf algebra map and H is understood as an \bar{H} -bicomodule with coactions $(\pi \otimes \text{id}) \circ \Delta_H$, $(\text{id} \otimes \pi) \circ \Delta_H$. Furthermore, \bar{A} is a right \bar{H} -comodule algebra and $B = \bar{A}^{\text{co } \bar{H}}$.

PROPOSITION 4.1. *The prolongation $\bar{A} \square_{\bar{H}} H$ is a cleft extension of B if and only if there exists a right \bar{H} -colinear, unital, convolution invertible map $f : H \rightarrow \bar{A}$.*

Proof. Given f define $\theta : \bar{A} \square_{\bar{H}} H \rightarrow B \otimes H$ by $\theta(a \otimes h) = af^{-1}(h_{(1)}) \otimes h_{(2)}$. Obviously θ is a left B -module map and a right H -comodule map. The convolution inverse f^{-1} satisfies the following covariance property:

$$(4.1) \quad f^{-1}(h_{(2)}) \otimes S(\pi(h_{(1)})) = f^{-1}(h)_{(0)} \otimes f^{-1}(h)_{(1)}.$$

This can be easily proven by standard Hopf algebraic techniques as follows. Consider the identity:

$$f^{-1}(h_{(1)})f(h_{(2)}) \otimes S(\pi(h_{(3)})) \otimes f^{-1}(h_{(4)}) = 1 \otimes S(\pi(h_{(1)})) \otimes f^{-1}(h_{(2)}).$$

Apply the \bar{H} -coaction to the first tensorand and use the facts that the coaction is an algebra map and that f is right \bar{H} -colinear to obtain:

$$\begin{aligned} f^{-1}(h_{(1)})_{(0)}f(h_{(2)}) \otimes f^{-1}(h_{(1)})_{(1)}\pi(h_{(3)}) \otimes S(\pi(h_{(4)})) \otimes f^{-1}(h_{(5)}) \\ = 1 \otimes 1 \otimes S(\pi(h_{(1)})) \otimes f^{-1}(h_{(2)}). \end{aligned}$$

Multiply the middle legs and use the assumption that π is a Hopf algebra map to derive the following equality:

$$f^{-1}(h_{(1)})_{(0)}f(h_{(2)}) \otimes f^{-1}(h_{(1)})_{(1)} \otimes f^{-1}(h_{(3)}) = 1 \otimes S(\pi(h_{(1)})) \otimes f^{-1}(h_{(2)}).$$

Finally multiplication of the first and the third tensorands yields the desired formula.

We use (4.1) to prove that the image of θ is in $B \otimes H$. Apply $\rho^{\bar{H}} \otimes H$ to $\theta(a \otimes h)$, where $a \otimes h \in \bar{A} \square_{\bar{H}} H$ (implicit suppressed summation). Then,

$$\begin{aligned} \rho^{\bar{H}}(af^{-1}(h_{(1)})) \otimes h_{(2)} &= a_{(0)}f^{-1}(h_{(1)})_{(0)} \otimes a_{(1)}f^{-1}(h_{(1)})_{(1)} \otimes h_{(2)} \\ &= a_{(0)}f^{-1}(h_{(2)}) \otimes a_{(1)}S(\pi(h_{(1)})) \otimes h_{(3)} = af^{-1}(h_{(3)}) \otimes \pi(h_{(1)})S(\pi(h_{(2)})) \otimes h_{(4)} \\ &= af^{-1}(h_{(1)}) \otimes 1 \otimes h_{(2)}, \end{aligned}$$

where the fact that $a \otimes h \in \bar{A} \square_{\bar{H}} H$ was used in the derivation of the third equality.

The inverse of θ is:

$$\theta^{-1} : B \otimes H \ni b \otimes h \longmapsto bf(h_{(1)}) \otimes h_{(2)} \in \bar{A} \square_{\bar{H}} H.$$

The right H -colinearity of f ensures that the map θ^{-1} has the specified codomain. A straightforward calculation that employs convolution-invertibility of f confirms that θ^{-1} is really the inverse of θ as claimed. Hence $B \otimes H$ is a cleft extension.

Conversely, suppose that $\bar{A} \square_{\bar{H}} H$ is a cleft extension. Then there exists a convolution invertible, normalized map $j \in \text{Hom}^H(H, \bar{A} \square_{\bar{H}} H)$. In view of the identification $\text{Hom}^H(H, \bar{A} \square_{\bar{H}} H) \simeq \text{Hom}^{\bar{H}}(H, \bar{A})$ the existence of a right H -colinear map j is equivalent

to the existence of the right \bar{H} -colinear map $f := (\text{id} \square_{\bar{H}} \varepsilon_H) \circ j$. Write $j(h) = h^{(1)} \otimes h^{(2)}$ so that $f(h) = h^{(1)} \varepsilon_H(h^{(2)})$. Since $j(1_H) = 1_{\bar{A}} \otimes 1_H$, $f(1_H) = 1_{\bar{A}}$. Write $j^{-1}(h) = h^{[1]} \otimes h^{[2]}$ for the convolution inverse of j . Then $f^{-1}(h) := h^{[1]} \varepsilon_H(h^{[2]})$ is the convolution inverse of f , since

$$\begin{aligned} f(h_{(1)})f^{-1}(h_{(2)}) &= h_{(1)}^{(1)} \varepsilon_H(h_{(1)}^{(2)}) h_{(2)}^{[1]} \varepsilon_H(h_{(2)}^{[2]}) = h_{(1)}^{(1)} h_{(2)}^{[1]} \varepsilon_H(h_{(1)}^{(2)} h_{(2)}^{[2]}) \\ &= \varepsilon_H(h) \varepsilon_H(1) = \varepsilon_H(h), \end{aligned}$$

as required. \square

In the setup of Proposition 4.1, the algebra structure on $B \otimes H$ induced by θ is:

$$\begin{aligned} (4.2) \quad (b \otimes h)(c \otimes g) &= \theta(\theta^{-1}(b \otimes h)\theta^{-1}(c \otimes g)) = \theta((bf(h_{(1)}) \otimes h_{(2)})(cf(g_{(1)}) \otimes g_{(2)})) \\ &= \theta(bf(h_{(1)})cf(g_{(1)}) \otimes h_{(2)}g_{(2)}) = bf(h_{(1)})cf(g_{(1)})f^{-1}(h_{(2)}g_{(2)}) \otimes h_{(3)}g_{(3)}, \end{aligned}$$

for all $b, c \in B$ and $g, h \in H$. We write $B \# H$ for the vector space $B \otimes H$ with this algebra structure.

Proposition 4.1 implies that if there exists a unital, convolution invertible and right \bar{H} -colinear map f , then the prolongation $\bar{A} \square_{\bar{H}} H$ is a principal H -comodule algebra. In terms of the map f , the standard strong connection for a cleft comodule algebra (2.2) comes out as

$$(4.3) \quad \ell(h) = f^{-1}(h_{(2)}) \otimes Sh_{(1)} \otimes f(h_{(3)}) \otimes h_{(4)} \in \bar{A} \square_{\bar{H}} H \otimes \bar{A} \square_{\bar{H}} H,$$

for all $h \in H$. If no further assumptions on π are made, we cannot yet conclude that \bar{A} is a principal \bar{H} -comodule algebra. More can be said if H is a left principal \bar{H} -comodule algebra.

COROLLARY 4.2. *Assume that π is onto and that H is a left principal \bar{H} -comodule algebra. If there exists a right \bar{H} -colinear, unital, convolution invertible map $f : H \rightarrow \bar{A}$, then \bar{A} is a principal right \bar{H} -comodule algebra.*

Proof. By Proposition 4.1, $\bar{A} \square_{\bar{H}} H$ is a principal H -comodule algebra, hence \bar{A} is a principal \bar{H} -comodule algebra by Lemma 2.4. \square

In some situations a strong connection in \bar{A} can be explicitly written down. The following corollary discusses the case applicable to quantum spheres and real projective spaces recalled in Section 3.

COROLLARY 4.3. *Assume that there exists a right \bar{H} -colinear, unital, convolution invertible map $f : H \rightarrow \bar{A}$. If there exists an \bar{H} bi-colinear section $\iota : \bar{H} \rightarrow H$ of π such that $\iota(1_{\bar{H}}) = 1_H$, then*

$$(4.4) \quad \ell : \bar{H} \longrightarrow \bar{A} \otimes \bar{A}, \quad h \longmapsto f^{-1}(\iota(h)_{(1)}) \otimes f(\iota(h)_{(2)}),$$

is a strong connection. Consequently, \bar{A} is a principal \bar{H} -comodule algebra.

Proof. Since in this case H is a principal left \bar{H} -comodule algebra [11, Corollary 2.6], the existence of a strong connection is indeed a corollary of Proposition 4.1 and Lemma 2.4.

Properties (2.1) can be checked directly as follows. By definition, for any $h \in \bar{H}$, $\pi(\iota(h)) = h$, therefore $\varepsilon_{\bar{H}}(\pi(\iota(h))) = \varepsilon_{\bar{H}}(h)$. Then,

$$(4.5) \quad \varepsilon_H(\iota(h)) = \varepsilon_{\bar{H}}(h),$$

since π is a Hopf algebra map. It follows that, for all $h \in \bar{H}$,

$$\ell(1) = f^{-1}(\iota(1_{\bar{H}})_{(1)}) \otimes f(\iota(1_{\bar{H}})_{(2)}) = f^{-1}(1_{H(1)}) \otimes f(1_{H(2)}) = 1_{\bar{A}} \otimes 1_{\bar{A}},$$

where we used the unitality of f , f^{-1} and ι . Using eq. (4.5) we obtain

$$f^{-1}(\iota(h)_{(1)})f(\iota(h)_{(2)}) = \varepsilon_H(\iota(h)) = \varepsilon_{\bar{H}}(h),$$

i.e. (2.1b) holds. To prove the right covariance (2.1c) of ℓ we use the right H -colinearity of f and the right \bar{H} -colinearity of ι :

$$\begin{aligned} \ell(h_{(1)}) \otimes h_{(2)} &= f^{-1}(\iota(h_{(1)})_{(1)}) \otimes f(\iota(h_{(1)})_{(2)}) \otimes h_{(2)} \\ &= f^{-1}(\iota(h)_{(1)}) \otimes f(\iota(h)_{(2)}) \otimes \pi(\iota(h)_{(3)}) = f^{-1}(\iota(h)_{(1)}) \otimes f(\iota(h)_{(2)})_{(0)} \otimes f(\iota(h)_{(2)})_{(1)}. \end{aligned}$$

Finally, using eq. (4.1) we obtain

$$\begin{aligned} f^{-1}(\iota(h)_{(1)})_{(1)} \otimes f^{-1}(\iota(h)_{(1)})_{(0)} \otimes f(\iota(h)_{(2)}) \\ = S(\pi(\iota(h)_{(1)})) \otimes f^{-1}(\iota(h)_{(2)}) \otimes f(\iota(h)_{(3)}) = S(h_{(1)}) \otimes f^{-1}(\iota(h_{(2)})_{(1)}) \otimes f(\iota(h_{(2)})_{(2)}), \end{aligned}$$

where, in the derivation of the last equality, we used the left colinearity of ι . This proves the left \bar{H} -colinearity of ℓ , eq. (2.1d). \square

Since a principal H -comodule algebra A is in particular a Hopf-Galois extension of coinvariants B (see Remark 2.2), the Hopf algebra H acts from the right on the centraliser subalgebra and H -subcomodule of A , $Z_A(B) := \{a \in A \mid ab = ba, \forall b \in B\}$, by the *Miyashita-Ulbrich action* [21]. Write $h^{[1]} \otimes_B h^{[2]}$ for the image of $h \in H$ under the translation map (see Remark 2.2). Then, for all $a \in Z_A(B)$, the Miyashita-Ulbrich action is given by

$$a \triangleleft h := h^{[1]}ah^{[2]}.$$

On the other hand, ${}^{\text{co}\bar{H}}H$ is a right H -coideal and a right H -module by the adjoint coaction,

$$k \triangleleft h := S(h_{(1)})kh_{(2)}, \quad \text{for all } k \in {}^{\text{co}\bar{H}}H, h \in H.$$

The following proposition is related to the (full) Hopf-Galois Reduction Theorem [8, 19, 12] if H is a left principal \bar{H} -comodule algebra. Namely, the map Ψ gives (in accordance with the Hopf-Galois Reduction Theorem) the reduction of $\bar{A} \square_{\bar{H}} H$ to \bar{A} .

PROPOSITION 4.4. *If there exists a right \bar{H} -colinear, unital, convolution invertible map $f : H \rightarrow \bar{A}$, then the following map:*

$$(4.6) \quad \Psi : {}^{\text{co}\bar{H}}H \longrightarrow Z_{B\#H}(B), \quad k \longmapsto f^{-1}(k_{(1)}) \otimes k_{(2)},$$

is an algebra, right H -colinear and right H -linear map.

Proof. Applying $\rho^{\bar{H}} \otimes \text{id}$ to $f^{-1}(k_{(1)}) \otimes k_{(2)}$ and using (4.1) and that $k \in {}^{\text{co}\bar{H}}H$ one immediately finds that $\Psi(k) \in B \otimes H$. The product in the algebra $B\#H$ is given by (4.2), hence, for all $b \in B$, $k \in {}^{\text{co}\bar{H}}H$,

$$(f^{-1}(k_{(1)}) \otimes k_{(2)})(b \otimes 1) = f^{-1}(k_{(1)})f(k_{(2)})bf^{-1}(k_{(3)}) \otimes k_{(4)} = bf^{-1}(k_{(1)}) \otimes k_{(2)},$$

i.e. $\Psi(k) \in Z_{B\#H}(B)$. Using (4.2) one easily finds that Ψ is an algebra map. It is clearly a right H -colinear map. To prove the right H -linearity of Ψ , first note that, for all $k \in {}^{\text{co } \bar{H}}H$,

$$\theta^{-1} \circ \Psi(k) = 1 \otimes k,$$

where $\theta^{-1} : B\#H \rightarrow \bar{A}\square_{\bar{H}}H$ is the isomorphism constructed in the proof of Proposition 4.1. The Miyashita-Ulbrich action can be calculated with the help of the strong connection (4.3)

$$\begin{aligned} (1 \otimes k) \triangleleft h &= (f^{-1}(h_{(2)}) \otimes Sh_{(1)})(1 \otimes k)(f(h_{(3)}) \otimes h_{(4)}) \\ &= f^{-1}(h_{(2)})f(h_{(3)}) \otimes Sh_{(1)}kh_{(4)} = 1 \otimes k \triangleleft h = \theta^{-1} \circ \Psi(k \triangleleft h). \end{aligned}$$

Therefore, Ψ is a right H -module map as required. \square

5. PROLONGATIONS OF QUANTUM SPHERES (OVER QUANTUM REAL PROJECTIVE SPACES)

5.1. Prolongations to $SU_q(2)$ -bundles. As the first illustration of Proposition 4.1 we consider the prolongation of principal $\mathcal{O}(\mathbb{Z}_2)$ -comodule algebras $\mathcal{O}(S_q^m)$ to principal $\mathcal{O}(SU_q(2))$ -comodule algebras along the Hopf algebra map $\pi : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(\mathbb{Z}_2)$ (3.8). As explained in Section 3, the coordinate algebras of quantum spheres form a hierarchy. Since the algebra maps in (3.4) are also right $\mathcal{O}(\mathbb{Z}_2)$ -colinear, the prolongation gives rise to the following hierarchy of principal $\mathcal{O}(SU_q(2))$ -comodule algebras:

$$(5.1) \quad \mathcal{O}(S_q^{m+1})\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2)) \xrightarrow{f_m \otimes \text{id}} \mathcal{O}(S_q^m)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2)), \quad m = 2, 3, \dots$$

Now, the Hopf algebra map $\pi : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(\mathbb{Z}_2)$ has a unital bilinear splitting ι , given in (3.9). This, in particular, implies that all the maps in (5.1) are surjective ($\mathcal{O}(SU_q(2))$ is coflat as a left $\mathcal{O}(\mathbb{Z}_2)$ -comodule). Furthermore we can invoke the reduction theorem to conclude that each of the principal $\mathcal{O}(\mathbb{Z}_2)$ -comodule algebras $\mathcal{O}(S_q^m)$ is a reduction of the principal $\mathcal{O}(SU_q(2))$ -comodule algebra $\mathcal{O}(S_q^m)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$. The coaction invariants of both $\mathcal{O}(S_q^m)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$ and $\mathcal{O}(S_q^m)$ necessarily coincide with $\mathcal{O}(\mathbb{RP}_q^m)$ (see (2.3)).

Since $\mathcal{O}(S_q^3) = \mathcal{O}(SU_q(2))$ more can be said about the last two algebras in hierarchy (5.1). First, $f_2 : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(S_q^2)$ (see (3.6)) is a right $\mathcal{O}(\mathbb{Z}_2)$ -colinear algebra map. As an algebra map, f_2 is convolution invertible with the (convolution) inverse $f_2^{-1} = f_2 \circ S$, i.e.

$$f_2^{-1} : a \mapsto z_0^*, \quad b \mapsto -q^{-1}z_1, \quad a^* \mapsto z_0, \quad b^* \mapsto -qz_1.$$

Therefore, $\mathcal{O}(S_q^2)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$ is a cleft (in fact trivial) principal comodule algebra by Proposition 4.1.

Since f_2 is an algebra map, there is a left action of $\mathcal{O}(SU_q(2))$ on $\mathcal{O}(\mathbb{RP}_q^2)$,

$$(5.2) \quad h \triangleright x = f_2(h_{(1)})xf_2(S(h_{(2)})),$$

and the algebra structure of $\mathcal{O}(S_q^2)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$ calculated from (4.2) is that of the smash product $\mathcal{O}(\mathbb{RP}_q^2)\# \mathcal{O}(SU_q(2))$,

$$(x \otimes h)(y \otimes h') = xh_{(1)} \triangleright y \otimes h_{(2)}h', \quad \text{for all } x, y \in \mathcal{O}(\mathbb{RP}_q^2), h, h' \in \mathcal{O}(SU_q(2)).$$

Using the explicit form of f_2 one easily derives the formulae for the action of generators of $\mathcal{O}(SU_q(2))$ on generators of $\mathcal{O}(\mathbb{RP}_q^2)$:

$$\begin{aligned} a \triangleright P &= q^2 P + q^2(1 - q^2)P^2, & b \triangleright P &= q(1 - q^2)PT, & a \triangleright R &= R + q^4(1 - q^2)PR, \\ b \triangleright R &= q(1 - q^2)TR, & a \triangleright T &= qT + q^3(1 - q^2)PT, & b \triangleright T &= (1 - q^2)T^2. \end{aligned}$$

Thus we can conclude that the *non-trivial* principal $\mathcal{O}(\mathbb{Z}_2)$ -comodule algebra $\mathcal{O}(S_q^2)$ is a reduction of the *trivial* principal $\mathcal{O}(SU_q(2))$ -comodule algebra $\mathcal{O}(\mathbb{RP}_q^2) \# \mathcal{O}(SU_q(2))$ (both over $\mathcal{O}(\mathbb{RP}_q^2)$).

The penultimate algebra in hierarchy (5.1), $\mathcal{O}(S_q^3) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(SU_q(2))$, is a trivial principal comodule algebra since the identity map $\mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(S_q^3)$ fulfills all the assumptions of Proposition 4.1.

5.2. Prolongations to $U(1)$ -bundles. One can use Hopf algebra map $\pi_2 : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(\mathbb{Z}_2)$ (3.7) to construct prolongations of the $\mathcal{O}(S_q^m)$ to principal $\mathcal{O}(U(1))$ -comodule algebras. The sequence of algebra maps (3.4) now yields a sequence of algebra maps between principal $\mathcal{O}(U(1))$ -comodule algebras

$$(5.3) \quad \mathcal{O}(S_q^{m+1}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1)) \xrightarrow{f_m \otimes \text{id}} \mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1)), \quad m = 2, 3, \dots$$

Since π_2 is split by a bicomodule map $u \mapsto v$ and each of the f_m is surjective, also all the maps in (5.3) are surjective.

PROPOSITION 5.1. *For all natural numbers $m > 1$, the principal $\mathcal{O}(U(1))$ -comodule algebras $\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ are non-trivial.*

Proof. This can be proven by induction. If $m = 2, 3$, then non-zero multiples of the identity are the only units in $\mathcal{O}(S_q^m)$ [11]. Since $\mathcal{O}(U(1))$ is the algebra of Laurent polynomials in one variable and the generator of $\mathcal{O}(U(1))$ is a group-like element, any convolution invertible map $f : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(S_q^m)$ must have the form $f(v) = \lambda 1$, $\lambda \in \mathbb{C}^\times$. Such a map cannot be right $\mathcal{O}(\mathbb{Z}_2)$ -colinear, as the coactions send v to $v \otimes u$ and 1 to $1 \otimes 1$. Therefore, there are no right $\mathcal{O}(\mathbb{Z}_2)$ -colinear convolution invertible maps $\mathcal{O}(U(1)) \rightarrow \mathcal{O}(S_q^m)$, and $\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ is non-cleft (hence non-trivial) by Proposition 4.1 if $m = 2, 3$.

Take $m > 1$ for which there are no convolution invertible $\mathcal{O}(\mathbb{Z}_2)$ -colinear maps from $\mathcal{O}(U(1))$ to $\mathcal{O}(S_q^m)$. Suppose there exists a convolution invertible, $\mathcal{O}(\mathbb{Z}_2)$ -colinear map $f : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(S_q^{m+1})$. Then $f_m \circ f$ would be a convolution invertible, $\mathcal{O}(\mathbb{Z}_2)$ -colinear map from $\mathcal{O}(U(1))$ to $\mathcal{O}(S_q^m)$ contradicting the inductive assumption. \square

PROPOSITION 5.2. (1) *For all integers $n \geq 1$, $\mathcal{O}(S_q^{2n+1}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ is an algebra isomorphic to $\mathcal{O}(S_q^{2n+1}) \otimes \mathcal{O}(U(1))$.*

(2) *For all integers $n > 1$, $\mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ is a subalgebra of $\mathcal{O}(S_q^{2n}) \otimes \mathcal{O}(U(1))$ isomorphic to a polynomial $*$ -algebra \mathcal{A}^{2n} generated by $\zeta_0, \zeta_1, \dots, \zeta_n$ and a central unitary ξ subject to the following relations*

$$(5.4a) \quad \zeta_i \zeta_j = q \zeta_j \zeta_i \quad \text{for } i < j, \quad \zeta_i \zeta_j^* = q \zeta_j^* \zeta_i \quad \text{for } i \neq j,$$

$$(5.4b) \quad \zeta_i \zeta_i^* = \zeta_i^* \zeta_i + (q^{-2} - 1) \sum_{m=i+1}^n \zeta_m \zeta_m^*, \quad \sum_{m=0}^n \zeta_m \zeta_m^* = 1, \quad \zeta_n^* = \zeta_n \xi.$$

Proof. The proof of part (1) is based on the following

LEMMA 5.3. *Let H be a commutative Hopf algebra, let $\pi : H \rightarrow \bar{H}$ be a Hopf algebra map, and let A be a right H -comodule algebra with coaction ρ^H . View A as a right \bar{H} -comodule algebra by the coaction $\rho^{\bar{H}} = (\text{id} \otimes \pi) \circ \rho^H$ and H as a left \bar{H} -comodule algebra by the coaction $(\pi \otimes \text{id}) \circ \Delta_H$. Then the algebra $A \square_{\bar{H}} H$ is isomorphic with $A \otimes {}^{co\bar{H}}H$.*

Proof. For all $a \in A$, write $a_{(0)} \otimes a_{(1)}$ for $\rho^H(a)$. Then the isomorphism is

$$\kappa : A \otimes {}^{co\bar{H}}H \longrightarrow A \square_{\bar{H}} H, \quad a \otimes h \longmapsto a_{(0)} \otimes a_{(1)} h,$$

with the inverse

$$\kappa^{-1} : A \square_{\bar{H}} H \longrightarrow A \otimes {}^{co\bar{H}}H, \quad \sum_i a^i \otimes h^i \longmapsto \sum_i a^i_{(0)} \otimes S(a^i_{(1)}) h^i.$$

This can be checked by a straightforward calculation. We only mention that the commutativity of H ensures that κ is an algebra map, and that, in general, κ is not an isomorphism of H -comodule algebras (it does not respect the obvious coactions obtained by restrictions of $\text{id} \otimes \Delta_H$). \square

Observe that each of the odd-dimensional spheres is a right $\mathcal{O}(U(1))$ -comodule algebra with the coaction $z_i \mapsto z_i \otimes v$. The $\mathcal{O}(\mathbb{Z}_2)$ coaction is obtained from the $\mathcal{O}(U(1))$ -coaction by applying the Hopf algebra map π_2 (3.7). Since $\mathcal{O}(U(1))$ is a commutative algebra, Lemma 5.3 can be applied and we deduce that $\mathcal{O}(S_q^{2n+1}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ is an algebra isomorphic to $\mathcal{O}(S_q^{2n+1}) \otimes {}^{co\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$. Note next that $\mathcal{O}(U(1)) = \mathbb{C}[v, v^*]$ is the algebra of Laurent polynomials (v^* is the inverse of v) and the coaction invariant subalgebra consists of all combinations of monomials of even degree, i.e. ${}^{co\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1)) = \mathbb{C}[v^2, v^{*2}]$. The latter is again the algebra of Laurent polynomials in one variable, hence it is isomorphic with $\mathcal{O}(U(1))$. This completes the proof of part (1).

(2) The vector space $\mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ is spanned by elements $x \otimes v^k$, where x is a monomial in $\mathcal{O}(S_q^{2n})$ with degree congruent to k modulo 2. All such elements can be generated by multiplying

$$(5.5) \quad \zeta_i = z_i \otimes v, \quad \xi = 1 \otimes v^{*2},$$

and their conjugates. Clearly $1 \otimes v^{*2}$ is central and unitary, the relations between the z_i are inherited from relations (3.1). In this way we obtain all but the last of relations (5.4). Since z_n is self-adjoint,

$$\zeta_n^* = z_n \otimes v^* = \zeta_n \xi.$$

This implies that equations (5.5) define a surjective $*$ -algebra map Φ from the $*$ -algebra \mathcal{A}^{2n} generated by ζ_i, ξ and relations (5.4) to $\mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$. Employing Diamond Lemma, one easily finds that a basis for \mathcal{A}^{2n} consists of monomials

$$\zeta_0^k \zeta_1^{k_1} \cdots \zeta_n^{k_n} \zeta_1^{*l_1} \cdots \zeta_{n-1}^{*l_{n-1}} \xi^m, \quad k, m \in \mathbb{Z}, \quad k_i, l_i \in \mathbb{N},$$

where, by convention, $\zeta_0^{-|k|}$ denotes $\zeta_0^{*|k|}$. Since all these elements are mapped by Φ to linearly independent vectors

$$z_0^k z_1^{k_1} \cdots z_n^{k_n} z_1^{*l_1} \cdots z_{n-1}^{*l_{n-1}} \otimes v^{\sum_i k_i - \sum_i l_i + k - 2m}, \quad k, m \in \mathbb{Z}, \quad k_i, l_i \in \mathbb{N},$$

in $\mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$, Φ is an injective $*$ -algebra map as required. \square

Let us make a few comments on the representation theory of algebras constructed in Proposition 5.2. Due to the close relationship of these algebras with coordinate algebras of quantum spheres, their representation theory bears close resemblance to that of the latter (see e.g. [14, Section 3] for the detailed discussion of representations of quantum spheres). We concentrate on the (more interesting) algebras in part (2) of Proposition 5.2. Non-equivalent irreducible $*$ -representations of $\mathcal{A}^{2n} \simeq \mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ split into two classes depending on whether ζ_n is represented by a non-zero or the zero operator. First, there is a family of representations labeled by $\varphi \in (0, 2\pi)$ and the sign \pm . For each φ and \pm the representation space $V_{\varphi, \pm}$ of $\pi_{\varphi, \pm}$ has an orthonormal basis: $|k_0, k_1, \dots, k_{n-1}\rangle$, $k_i = 0, 1, 2, \dots$. On this basis of $V_{\varphi, \pm}$, the (bounded) operators representing ζ_i, ξ act as follows:

$$(5.6a) \quad \pi_{\varphi, \pm}(\zeta_n)|k_0, \dots, k_{n-1}\rangle = \pm e^{i\varphi} q^{k_0 + \dots + k_{n-1} + n} |k_0, \dots, k_{n-1}\rangle,$$

$$(5.6b) \quad \pi_{\varphi, \pm}(\zeta_l)|k_0, \dots, k_{n-1}\rangle = (1 - q^{2k_l})^{1/2} q^{k_0 + \dots + k_{l-1} + l} |k_0, \dots, k_l - 1, \dots, k_{n-1}\rangle, \quad l < n,$$

$$(5.6c) \quad \pi_{\varphi, \pm}(\xi)|k_0, \dots, k_{n-1}\rangle = e^{-2i\varphi} |k_0, \dots, k_{n-1}\rangle.$$

Second, if ζ_n is represented by the zero operator, then the S_q^{2n} -sphere part of the algebra $\mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ collapses to the odd-dimensional quantum sphere S_q^{2n-1} , and we are essentially in the situation described in part (1) of Proposition 5.2. Therefore, representations of \mathcal{A}^{2n} are given by the tensor product of those for the odd-dimensional quantum sphere S_q^{2n-1} and the circle group $U(1)$. Explicitly, there is a family of representations $\pi_{\lambda, \mu}$ labeled by λ, μ such that $|\lambda| = |\mu| = 1$. The orthonormal basis for the corresponding representation space $V_{\lambda, \mu}$ is $|k_0, k_1, \dots, k_{n-1}\rangle$, $k_i = 0, 1, 2, \dots$. On this basis of $V_{\lambda, \mu}$, the (bounded) operators representing ζ_i, ξ act as follows:

$$(5.7a) \quad \pi_{\lambda, \mu}(\zeta_n) = 0, \quad \pi_{\lambda, \mu}(\zeta_{n-1})|k_0, \dots, k_{n-1}\rangle = \lambda q^{k_0 + \dots + k_{n-1} + n} |k_0, \dots, k_{n-1}\rangle,$$

$$(5.7b)$$

$$\pi_{\lambda, \mu}(\zeta_l)|k_0, \dots, k_{n-1}\rangle = (1 - q^{2k_l})^{1/2} q^{k_0 + \dots + k_{l-1} + l} |k_0, \dots, k_l - 1, \dots, k_{n-1}\rangle, \quad l < n - 1,$$

$$(5.7c) \quad \pi_{\lambda, \mu}(\xi)|k_0, \dots, k_{n-1}\rangle = \mu |k_0, \dots, k_{n-1}\rangle.$$

In the case of algebras $\mathcal{O}(S_q^{2n-1}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$, by the part (1) of Proposition 5.2 irreducible representations have the form (5.7) (without the first of equations (5.7a)), where the ζ_l should be replaced by the z_l and ξ corresponds to the unitary generator of $\mathcal{O}(U(1))$.

The representations $\pi_{\varphi, \pm}$ can be used to construct Fredholm modules [5, Chapter 4] over the algebras $\mathcal{A}^{2n} \simeq \mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ in the same way Fredholm modules over coordinate algebras of even quantum spheres are constructed in [14]. For each φ , an even Fredholm module $(\mathcal{H}_\varphi, F, \gamma)$ is given by the representation

$$\pi_\varphi = \pi_{\varphi, +} \oplus \pi_{\varphi, -} \quad \text{on} \quad \mathcal{H}_\varphi = V_{\varphi, +} \oplus V_{\varphi, -},$$

with operators

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Obviously F is self-adjoint, squares to 1 and anti-commutes with γ , as required. As in [14, Section 4.1.1], for all $a \in \mathcal{O}(S_q^{2n}) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$,

$$[F, \pi_\varphi(a)] = \begin{pmatrix} 0 & -\pi_{\varphi,+}(a) + \pi_{\varphi,-}(a) \\ \pi_{\varphi,+}(a) - \pi_{\varphi,-}(a) & 0 \end{pmatrix}.$$

Note that $\pi_{\varphi,+} - \pi_{\varphi,-} = \pi_{\varphi,+} \circ (\text{id} - \nu)$, where ν is the $*$ -algebra automorphism of \mathcal{A}^{2n} given by

$$\nu : \xi \mapsto \xi, \quad \zeta_i \mapsto \begin{cases} \zeta_i & \text{if } i \neq n \\ -\zeta_n & \text{if } i = n \end{cases}.$$

This implies that $\pi_{\varphi,+}(a) - \pi_{\varphi,-}(a)$ is always a multiple of $\pi_{\varphi,+}(\zeta_n)$, which is a compact, trace-class operator. Therefore, $[F, \pi_\varphi(a)]$ is a compact, in fact trace-class, operator on \mathcal{H}_φ . Thus $(\mathcal{H}_\varphi, F, \gamma)$ is a 1-summable Fredholm module over \mathcal{A}^{2n} . This allows one to define a trace τ or the zero-component of the Chern character of $(\mathcal{H}_\varphi, F, \gamma)$ by

$$\tau(a) := \text{Tr}(\gamma \pi_\varphi(a)) = \text{Tr}(\pi_{\varphi,+}(a) - \pi_{\varphi,-}(a)).$$

On the basis of \mathcal{A}^{2n} given in the proof of Proposition 5.2 (2), $\zeta_0^{k_0} \zeta_1^{k_1} \dots \zeta_n^{k_n} \zeta_1^{*l_1} \dots \zeta_{n-1}^{*l_{n-1}} \xi^m$, the trace is non-zero (and given by a rational function of q multiplied by $e^{i(k_n - 2m)\varphi}$) only when simultaneously k_n is odd, $k = 0$ and $k_i = l_i$, for all $i < n$.

REMARK 5.4. Lemma 5.3 can also be used to determine the algebras obtained as prolongations of the $\mathcal{O}(\mathbb{Z}_p)$ -coaction on $\mathcal{O}(S_q^{2n+1})$, for all $p > 1$. $\mathcal{O}(\mathbb{Z}_p)$ is a Hopf $*$ -algebra generated by w subject to relations $w^p = 1$ and $w^* = w^{p-1}$. The coaction of $\mathcal{O}(\mathbb{Z}_p)$ on $\mathcal{O}(S_q^{2n+1})$ is given on generators by $z_i \mapsto z_i \otimes w$. The subalgebra of coaction invariants is known as the *quantum lens space* $\mathcal{O}(L_q(p; \mathbf{1}))$ [16]. The coaction of $\mathcal{O}(\mathbb{Z}_p)$ on $\mathcal{O}(S_q^{2n+1})$ can be equivalently defined as the projection of the $\mathcal{O}(U(1))$ -coaction through the Hopf $*$ -algebra map $\pi_p : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(\mathbb{Z}_p)$, $v \mapsto w$. Now, the combination of Lemma 5.3 with the arguments of the proof of Proposition 5.2 (1) yields a $*$ -algebra isomorphism

$$\mathcal{O}(S_q^{2n+1}) \square_{\mathcal{O}(\mathbb{Z}_p)} \mathcal{O}(U(1)) \simeq \mathcal{O}(S_q^{2n+1}) \otimes \mathcal{O}(U(1)).$$

◇

The Hopf algebra map $f_1 \circ f_2 : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{O}(U(1))$, given on generators by $a \mapsto v$, $b \mapsto 0$ (see Section 3) has a unital $\mathcal{O}(U(1))$ -bilinear splitting [4, p. 200], [11, p. 257]

$$\iota : v^n \mapsto a^n, \quad v^{*n} \mapsto a^{*n}.$$

This implies that $\mathcal{O}(SU_q(2))$ is a left principal $\mathcal{O}(U(1))$ -comodule algebra. By the identification

$$\begin{aligned} \mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(SU_q(2)) &\simeq \mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} (\mathcal{O}(U(1)) \square_{\mathcal{O}(U(1))} \mathcal{O}(SU_q(2))) \\ &\simeq (\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))) \square_{\mathcal{O}(U(1))} \mathcal{O}(SU_q(2)), \end{aligned}$$

principal comodule algebras in (5.3) can also be understood as reductions of those in (5.1) by the Hopf ideal $\ker(f_1 \circ f_2) \subseteq \mathcal{O}(SU_q(2))$. In particular, for $m = 2, 3$, the algebras $\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(U(1))$ are *non-trivial* principal $\mathcal{O}(U(1))$ -comodule algebras obtained as reductions of *trivial* principal $\mathcal{O}(SU_q(2))$ -comodule algebras $\mathcal{O}(S_q^m) \square_{\mathcal{O}(\mathbb{Z}_2)} \mathcal{O}(SU_q(2))$.

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